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Q-polynomial distance-regular graphs with $a_1 = 0$ and $a_2 \neq 0$

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Abstract

Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Let X denote the vertex set of Γ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix $x \in X$ and let $A^* \in \mathrm{Mat}_X(\mathbb{C})$ denote the corresponding dual adjacency matrix. Let T denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^*. We call T the *Terwilliger algebra* of Γ with respect to x. We show that up to isomorphism there exists a unique irreducible T-module W with endpoint 1. We show that W has dimension $2D - 2$. We display a basis for W which consists of eigenvectors for A^*. We display the action of A on this basis. We show that W appears in the standard module of Γ with multiplicity $k - 1$, where k is the valency of Γ.

1. Introduction

Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and intersection numbers a_i, b_i, c_i (see Section 2 for formal definitions). We recall the Terwilliger algebra of Γ. Let X denote the vertex set of Γ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix a "base vertex" $x \in X$ and let $A^* \in \mathrm{Mat}_X(\mathbb{C})$ denote the corresponding dual adjacency matrix. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^*. The algebra T is called the *Terwilliger algebra* of Γ with respect to x [26]. T is closed under the conjugate-transpose map so T is semi-simple [26, Lemma 3.4(i)]. Therefore each T-module is a direct sum of irreducible T-modules. Describing the irreducible T-modules is an active area of research [3–17,25–27].

In this description there is an important parameter called the *endpoint* which we now recall. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i \mid 0 \leq i \leq$

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$D, \; E_i^*W \neq 0\}$, where $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ is the projection onto the ith subconstituent of Γ with respect to x [26, p. 378]. There exists a unique irreducible T-module with endpoint 0 [12, Proposition 8.4]; for a detailed description see [7,12]. In this paper we consider irreducible T-modules with endpoint 1.

We are going to make an assumption about the intersection numbers of Γ. In order to motivate things let us review the situation for $a_1 = 0$. In this case either (i) $a_i = 0$ for $0 \le i \le D$ (the bipartite case); or (ii) $a_i = 0$ for $0 \le i \le D-1$ and $a_D \neq 0$ (the almost bipartite case); or (iii) $a_i \neq 0$ for $2 \le i \le D$ [21, Theorem 6.3]. In case (i) (resp. case (ii)) the irreducible T-modules were determined in [7–9] (resp. [4,6]). In this paper we assume case (iii) and describe the irreducible T-modules with endpoint 1. Our results are summarized as follows. We show that up to isomorphism there exists a unique irreducible T-module W with endpoint 1. We show that W has dimension $2D-2$. We display a basis for W which consists of eigenvectors of A^*. We display the action of A on this basis. We show that W appears in the standard module of Γ with multiplicity $k-1$, where k is the valency of Γ. We remark that the following Q-polynomial distance-regular graphs satisfy $a_1 = 0$ and $a_i \neq 0$ for $2 \le i \le D$: the Hermitian forms graphs with $r = 2$ [2, Section 9.5.C] and the Witt graph M_{23} [2, Section 11.4.B]. We discuss these examples later in the paper.

2. Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitian inner product $\langle , \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$, where t denotes transpose and $^-$ denotes complex conjugation. For $y \in X$ let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following will be useful: for each $B \in \mathrm{Mat}_X(\mathbb{C})$ we have

$$\langle u, Bv\rangle = \langle \overline{B}^t u, v\rangle \quad (u, v \in V). \tag{1}$$

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For a vertex $x \in X$ and an integer $i \ge 0$ let $\Gamma_i(x)$ denote the set of vertices at distance i from x. We use the abbreviation $\Gamma(x) = \Gamma_1(x)$. For an integer $k \ge 0$ we say that Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ.

For the rest of this paper we assume that Γ is distance-regular with diameter $D \ge 3$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \le h, i, j \le D$. For convenience set $c_i := p^i_{1,i-1}$ $(1 \le i \le D)$, $a_i := p^i_{1i}$ $(0 \le i \le D)$, $b_i := p^i_{1,i+1}$ $(0 \le i \le D-1)$, $k_i := p^0_{ii}$ $(0 \le i \le D)$, and $c_0 = b_D = 0$.

By the triangle inequality the following hold for $0 \le h, i, j \le D$: (i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two; (ii) $p^h_{ij} \ne 0$ if one of h, i, j equals the sum of the other two. In particular $c_i \ne 0$ for $1 \le i \le D$ and $b_i \ne 0$ for $0 \le i \le D-1$. We observe that Γ is regular with valency $k = k_1 = b_0$ and that $c_i + a_i + b_i = k$ for $0 \le i \le D$. Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \le i \le D$. By [2, p. 127],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \le i \le D). \tag{2}$$

We recall the Bose–Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i \end{cases} \quad (x, y \in X). \tag{3}$$

We call A_i the ith *distance matrix* of Γ. We use the abbreviation $A := A_1$ and call this the *adjacency matrix* of Γ. We observe that (ai) $A_0 = I$; (aii) $\sum_{i=0}^D A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (aiv) $A_i^t = A_i$ $(0 \le i \le D)$; (av) $A_i A_j = \sum_{h=0}^D p^h_{ij} A_h$ $(0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find that $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose–Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [2, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $\sum_{i=0}^D E_i = I$; (eiii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (eiv) $E_i^t = E_i$ $(0 \le i \le D)$; (ev) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \le i, j \le D$, so M is closed under $\circ$. Thus there exist complex scalars $q^h_{ij} (0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \quad (0 \le i, j \le D).$$

By [2, Proposition 4.1.5], q^h_{ij} is real and nonnegative for $0 \le h, i, j \le D$. The q^h_{ij} are called the *Krein parameters of* Γ. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \le h, i, j \le D$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. From now on assume that Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We recall the dual eigenvalues of Γ. Define complex scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ by

$$E_1 = |X|^{-1} \sum_{i=0}^{D} \theta^*_i A_i.$$

We call the sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ the *dual eigenvalue sequence of* Γ for the given Q-polynomial structure. The scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are real and mutually distinct [26, Lemma 3.11(iii)].

3. The Terwilliger algebra

In this section we recall the dual Bose–Mesner algebra and the Terwilliger algebra of Γ. Fix a vertex $x \in X$. We view x as a "base vertex". For $0 \le i \le D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X). \tag{4}$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [26, p. 378]. We observe that (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^* (0 \le i \le D)$; (iii) $E_i^{*t} = E_i^* (0 \le i \le D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^* (0 \le i, j \le D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose–Mesner algebra* of Γ with respect to x [26, p. 378]. For $0 \le i \le D$ we have

$$E_i^* V = \mathrm{Span}\{\hat{y} \mid y \in X, \partial(x, y) = i\}$$

so $\dim E_i^* V = k_i$. We call $E_i^* V$ the ith *subconstituent of* Γ with respect to x. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}. \tag{5}$$

Moreover E_i^* is the projection from V onto $E_i^* V$ for $0 \le i \le D$.

Let $A^* = A^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$A^*_{yy} = |X| E_{xy} \quad (y \in X),$$

where $E = E_1$. By the construction, $A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. We call A^* the *dual adjacency matrix of* Γ with respect to x. By [26, Lemma 3.11(ii)], A^* generates M^*.

We recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the *Terwilliger algebra of* Γ with respect to x [26, Definition 3.3]. Recall that M (resp. M^*) is generated by A (resp. A^*) so T is generated by A, A^*. We observe that T has finite dimension. By construction T is closed under the conjugate-transpose map so T is semi-simple [26, Lemma 3.4(i)].

By a *T-module* we mean a subspace W of V such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Assume that W is irreducible. Then A and A^* act on W as a tridiagonal pair [17, Example 1.4]. We refer the reader to [17–20,22,23] and the references therein for background on tridiagonal pairs.

By [14, Corollary 6.2] any T-module is an orthogonal direct sum of irreducible T-modules. In particular the standard module V is an orthogonal direct sum of irreducible T-modules. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W, W' are said to be *isomorphic* whenever there exists an isomorphism of T-modules from W to W'. By [7, Lemma 3.3] any two nonisomorphic irreducible T-modules are orthogonal. Let W denote an irreducible T-module. By [26, Lemma 3.4(iii)] W is an orthogonal direct sum of the nonvanishing spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^* W \neq 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \le i \le D,\ E_i^* W \neq 0\}| - 1$.

By [12, Proposition 8.3, Proposition 8.4] $M\hat{x}$ is the unique irreducible T-module with endpoint 0 and the unique irreducible T-module with diameter D. Moreover $M\hat{x}$ is the unique irreducible T-module on which E_0 does not vanish. We call $M\hat{x}$ the *primary module*. From now until the end of Section 10 we adopt the following notational convention.

Notation 3.1. Let $\Gamma = (X, R)$ denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and valency k. We assume that $a_1 = 0$, $a_2 \neq 0$ and note that $a_i \neq 0$ for $2 \leq i \leq D$ [21, Theorem 6.3]. Let $A_0, A_1, \ldots, A_D$ denote the distance matrices of Γ, and let V denote the standard module of Γ. We fix $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, and $T = T(x)$ denote the corresponding dual idempotents and Terwilliger algebra, respectively. Let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the dual eigenvalue sequence of Γ for the given Q-polynomial structure.

4. The sets D^i_j

With reference to Notation 3.1, in this section we define certain subsets D^i_j of X and explore their properties.

Definition 4.1. With reference to Notation 3.1 fix $y \in \Gamma(x)$. For $0 \leq i, j \leq D$ we define $D^i_j = D^i_j(x, y)$ by

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(y).$$

For notational convenience we set $D^i_j = \emptyset$ if i or j is contained in $\{-1, D + 1\}$.

Lemma 4.2. *With reference to Notation 3.1 the following* (i), (ii) *hold for* $0 \leq i, j \leq D$.

(i) $|D^i_j| = p^1_{ij}$.
(ii) $D^i_j = \emptyset$ *if and only if* $p^1_{ij} = 0$.

Proof. (i) Immediate from the definition of p^1_{ij} and D^i_j.
(ii) Immediate from (i) above. ∎

Lemma 4.3 (*[2, p. 134]*). *With reference to Notation 3.1 the following* (i), (ii) *hold.*

(i) $p^1_{i-1,i} = p^1_{i,i-1} = c_i k_i k^{-1}$ $(1 \leq i \leq D)$.
(ii) $p^1_{ii} = a_i k_i k^{-1}$ $(0 \leq i \leq D)$.

Lemma 4.4. *With reference to Notation 3.1 the following* (i)–(iii) *hold.*

(i) $p^1_{i-1,i} \neq 0$, $p^1_{i,i-1} \neq 0$ $(1 \leq i \leq D)$.
(ii) $p^1_{00} = 0$, $p^1_{11} = 0$, $p^1_{ii} \neq 0$ $(2 \leq i \leq D)$.
(iii) $p^1_{ij} = 0$ *if* $|i - j| \notin \{0, 1\}$.

Proof. (i), (ii) Immediate from Lemma 4.3.
(iii) Immediate from the triangle inequality. ∎

Theorem 4.5 (*[21, Theorem 5.2]*). *With reference to Notation 3.1 fix* $y \in \Gamma(x)$ *and use the abbreviation* $D^i_j = D^i_j(x, y)$ *for* $0 \leq i, j \leq D$. *Then the following* (i), (ii) *hold.*

(i) *For* $1 \leq i \leq D$ *each* $z \in D^i_{i-1}$ *(resp.* D^{i-1}_i*) is adjacent to*

(a) *precisely*	ρ_i	*vertices in* D^{i-1}_{i-1},
(b) *precisely*	c_{i-1}	*vertices in* D^{i-1}_{i-2} *(resp.* D^{i-2}_{i-1}*)*,
(c) *precisely*	$c_i - c_{i-1} - \rho_i$	*vertices in* D^{i-1}_i *(resp.* D^i_{i-1}*)*,
(d) *precisely*	$a_{i-1} - \rho_i$	*vertices in* D^i_{i-1} *(resp.* D^{i-1}_i*)*,
(e) *precisely*	b_i	*vertices in* D^{i+1}_i *(resp.* D^i_{i+1}*)*,
(f) *precisely*	$a_i - a_{i-1} + \rho_i$	*vertices in* D^i_i,

where

$$\rho_i = a_{i-1}\frac{(\theta^*_0-\theta^*_{i-1})(\theta^*_2-\theta^*_i)-(\theta^*_1-\theta^*_{i-1})(\theta^*_1-\theta^*_i)}{(\theta^*_0-\theta^*_{i-1})(\theta^*_{i-1}-\theta^*_i)} \quad (2\le i\le D), \qquad \rho_1=0.$$

(ii) *For* $2\le i\le D$ *each* $z\in D^i_i$ *is adjacent to*

(a) *precisely* σ_i *vertices in* D^{i-1}_{i-1},
(b) *precisely* τ_i *vertices in* D^{i+1}_{i+1},
(c) *precisely* $c_i-\sigma_i$ *vertices in* D^{i}_{i-1},
(d) *precisely* $c_i-\sigma_i$ *vertices in* D^{i-1}_{i},
(e) *precisely* $b_i-\tau_i$ *vertices in* D^{i+1}_{i},
(f) *precisely* $b_i-\tau_i$ *vertices in* D^{i}_{i+1},
(g) *precisely* $a_i-b_i-c_i+\sigma_i+\tau_i$ *vertices in* D^i_i,

where

$$\sigma_i = c_i\frac{(\theta^*_0-\theta^*_{i})(\theta^*_2-\theta^*_i)-(\theta^*_1-\theta^*_{i-1})(\theta^*_1-\theta^*_i)}{(\theta^*_0-\theta^*_{i})(\theta^*_{i-1}-\theta^*_i)},$$

$$\tau_i = b_i\frac{(\theta^*_0-\theta^*_{i})(\theta^*_2-\theta^*_i)-(\theta^*_1-\theta^*_{i})(\theta^*_1-\theta^*_{i+1})}{(\theta^*_0-\theta^*_{i})(\theta^*_{i+1}-\theta^*_i)} \quad (2\le i\le D-1), \qquad \tau_D=0.$$

5. Some products in T

With reference to Notation 3.1, in this section we evaluate several products in T which we shall need later.

Lemma 5.1. *With reference to Notation 3.1, for* $0\le h,i,j\le D$ *and* $y,z\in X$ *the* (y,z)*-entry of* $E^*_hA_iE^*_j$ *is 1 if* $\partial(x,y)=h$, $\partial(y,z)=i$, $\partial(x,z)=j$, *and 0 otherwise.*

Proof. Compute the (y,z)-entry of $E^*_hA_iE^*_j$ by matrix multiplication and simplify the result using (3) and (4). ■

Corollary 5.2 (*[26, Lemma 3.2]*). *With reference to Notation 3.1,*

$$E^*_hA_iE^*_j=0 \quad \text{if and only if } p^h_{ij}=0 \quad (0\le h,i,j\le D).$$

Proof. Immediate from Lemma 5.1. ■

Corollary 5.3. *With reference to Notation 3.1 the following* (i)–(iii) *hold.*

(i) $E^*_1AE^*_1=0$.
(ii) $E^*_iA_{i-1}E^*_1+E^*_iA_iE^*_1+E^*_iA_{i+1}E^*_1=E^*_iJE^*_1$ *for* $1\le i\le D-1$.
(iii) $E^*_DA_{D-1}E^*_1+E^*_DA_DE^*_1=E^*_DJE^*_1$.

Proof. (i) By Corollary 5.2 and since $p^1_{11}=0$ by Lemma 4.4(ii).

(ii), (iii) For each equation evaluate the right-hand side using assertion (aii) below line (3), and simplify the result using Corollary 5.2 and assertion (i) above line (2). ■

Lemma 5.4. *With reference to Notation 3.1, for* $0\le h,i,j,r,s\le D$ *and* $y,z\in X$ *the* (y,z)*-entry of* $E^*_hA_rE^*_iA_sE^*_j$ *is* $|\Gamma_i(x)\cap\Gamma_r(y)\cap\Gamma_s(z)|$ *if* $\partial(x,y)=h$, $\partial(x,z)=j$, *and 0 otherwise.*

Proof. Compute the (y,z)-entry of $E^*_hA_rE^*_iA_sE^*_j$ by matrix multiplication and simplify the result using (3) and (4). ■

6. The matrices L, F, R

With reference to Notation 3.1, in this section we recall the matrices L, F, R and use them to interpret Theorem 4.5.

Definition 6.1. With reference to Notation 3.1 we define matrices $L = L(x)$, $F = F(x)$, $R = R(x)$ by

$$L = \sum_{h=1}^{D} E^*_{h-1} A E^*_h, \qquad F = \sum_{h=0}^{D} E^*_h A E^*_h, \qquad R = \sum_{h=0}^{D-1} E^*_{h+1} A E^*_h.$$

Note that $A = L + F + R$ [7, Lemma 4.4]. We call L, F, and R the *lowering matrix*, the *flat matrix*, and the *raising matrix of* Γ with respect to x, respectively.

Lemma 6.2. *With reference to Notation* 3.1 *and Definition* 6.1 *the following* (i)–(v) *hold.*

(i) $L E^*_1 = E^*_0 A E^*_1$.
(ii) $L E^*_2 A E^*_1 = b_1 E^*_1 + (c_2 - 1) E^*_1 A_2 E^*_1$.
(iii) $L E^*_i A_{i-1} E^*_1 = b_{i-1} E^*_{i-1} A_{i-2} E^*_1 + (b_{i-1} - \tau_{i-1}) E^*_{i-1} A_{i-1} E^*_1 + (c_i - c_{i-1} - \rho_i) E^*_{i-1} A_i E^*_1$ *for* $3 \le i \le D$.
(iv) $L E^*_2 A_2 E^*_1 = a_2 E^*_1 A_2 E^*_1$.
(v) $L E^*_i A_i E^*_1 = \tau_{i-1} E^*_{i-1} A_{i-1} E^*_1 + (a_i - a_{i-1} + \rho_i) E^*_{i-1} A_i E^*_1$ *for* $3 \le i \le D$.

Proof. For each equation and for $z, w \in X$ compute the (z, w)-entry of each side and interpret the results using Lemmas 5.1 and 5.4 and Theorem 4.5. ∎

Lemma 6.3. *With reference to Notation* 3.1 *and Definition* 6.1 *the following* (i)–(iv) *hold.*

(i) $F E^*_1 = 0$.
(ii) $F E^*_i A_{i-1} E^*_1 = (a_{i-1} - \rho_i) E^*_i A_{i-1} E^*_1 + (c_i - \sigma_i) E^*_i A_i E^*_1$ *for* $2 \le i \le D$.
(iii) $F E^*_i A_i E^*_1 = (a_i - a_{i-1} + \rho_i) E^*_i A_{i-1} E^*_1 + (a_i - b_i - c_i + \sigma_i + \tau_i) E^*_i A_i E^*_1 + \rho_{i+1} E^*_i A_{i+1} E^*_1$ *for* $2 \le i \le D - 1$.
(iv) $F E^*_D A_D E^*_1 = (a_D - a_{D-1} + \rho_D) E^*_D A_{D-1} E^*_1 + (a_D - c_D + \sigma_D) E^*_D A_D E^*_1$.

Proof. For each equation and for $z, w \in X$ compute the (z, w)-entry of each side and interpret the results using Lemmas 5.1 and 5.4 and Theorem 4.5. ∎

Lemma 6.4. *With reference to Notation* 3.1 *and Definition* 6.1 *the following* (i)–(iv) *hold.*

(i) $R E^*_i A_{i-1} E^*_1 = c_i E^*_{i+1} A_i E^*_1$ *for* $1 \le i \le D - 1$.
(ii) $R E^*_D A_{D-1} E^*_1 = 0$.
(iii) $R E^*_i A_i E^*_1 = \rho_{i+1} E^*_{i+1} A_i E^*_1 + \sigma_{i+1} E^*_{i+1} A_{i+1} E^*_1$ *for* $2 \le i \le D - 1$.
(iv) $R E^*_D A_D E^*_1 = 0$.

Proof. For each equation and for $z, w \in X$ compute the (z, w)-entry of each side and interpret the results using Lemmas 5.1 and 5.4 and Theorem 4.5. ∎

7. More products in T

With reference to Notation 3.1, in this section we evaluate more products in T which we will need later.

Lemma 7.1. *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and for $1 \le i \le D$ the number $|\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i-1}(z)|$ is equal to $c_i k_i k^{-1}$ if $y = z$ and $c_i(c_i - 1)k_i k^{-1} b_1^{-1}$ if $y \ne z$.*

Proof. Assume that $y \ne z$; otherwise the result follows by Lemma 4.3(i). Use the abbreviation $D_j^\ell = D_j^\ell(x, y)$ $(0 \le \ell, j \le D)$ and note that $z \in D_2^1$. It follows from Theorem 4.5 that the number of paths of length $i - 1$ between z and D_{i-1}^i is independent of x, y, z. Moreover, between any two vertices of Γ which are at distance $i - 1$, there exist exactly $c_1 c_2 \cdots c_{i-1}$ paths of length $i - 1$, implying that $|D_{i-1}^i \cap \Gamma_{i-1}(z)|$ is independent of x, y, z. For $v \in D_{i-1}^i$ we have $|\Gamma_{i-1}(v) \cap \Gamma(x)| = c_i$ so $|\Gamma_{i-1}(v) \cap D_2^1| = c_i - 1$. Counting the number of pairs (w, v) such that $w \in D_2^1$, $v \in D_{i-1}^i$, $\partial(w, v) = i - 1$ in two different ways and using the above comments we obtain $|D_{i-1}^i \cap \Gamma_{i-1}(z)||D_2^1| = |D_{i-1}^i|(c_i - 1)$. The result now follows by Lemma 4.2(i) and Lemma 4.3(i). ■

Corollary 7.2. *With reference to Notation 3.1, for $1 \le i \le D$ we have*

$$E_1^* A_{i-1} E_i^* A_{i-1} E_1^* = c_i k_i k^{-1} E_1^* + c_i(c_i - 1)k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

Proof. For $z, w \in X$ we show that the (z, w)-entries of both sides are equal. If $z \notin \Gamma(x)$ or $w \notin \Gamma(x)$ then the (z, w)-entry of each side is 0. If $z, w \in \Gamma(x)$ then the (z, w)-entries of both sides are equal by Lemmas 5.1, 5.4 and 7.1. The result follows. ■

Lemma 7.3. *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and for $1 \le i \le D$ the number $|\Gamma_i(x) \cap \Gamma_i(y) \cap \Gamma_{i-1}(z)|$ is equal to 0 if $y = z$ and $a_i c_i k_i k^{-1} b_1^{-1}$ if $y \ne z$.*

Proof. Assume that $y \ne z$; otherwise the result is clear. Use the abbreviation $D_j^\ell = D_j^\ell(x, y)$ $(0 \le \ell, j \le D)$ and note that $z \in D_2^1$. Similarly as in the proof of Lemma 7.1 we find that $|D_i^i \cap \Gamma_{i-1}(z)|$ is independent of x, y, z, and that $|\Gamma_{i-1}(v) \cap D_2^1| = c_i$ for $v \in D_i^i$. Counting the number of pairs (w, v) such that $w \in D_2^1$, $v \in D_i^i$, $\partial(w, v) = i - 1$ in two different ways and using the above comments we obtain $|D_i^i \cap \Gamma_{i-1}(z)||D_2^1| = |D_i^i| c_i$. The result now follows by Lemma 4.2(i) and Lemma 4.3. ■

Corollary 7.4. *With reference to Notation 3.1, for $1 \le i \le D$ we have*

$$E_1^* A_{i-1} E_i^* A_i E_1^* = a_i c_i k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

Proof. For $z, w \in X$ we show that the (z, w)-entries of both sides are equal. If $z \notin \Gamma(x)$ or $w \notin \Gamma(x)$ then the (z, w)-entry of each side is 0. If $z, w \in \Gamma(x)$ then the (z, w)-entries of both sides are equal by Lemmas 5.1, 5.4 and 7.3. The result follows. ■

Lemma 7.5. *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and for $1 \le i \le D$ the number $|\Gamma_i(x) \cap \Gamma_i(y) \cap \Gamma_i(z)|$ is equal to $a_i k_i k^{-1}$ if $y = z$ and $a_i(a_i - 1)k_i k^{-1} b_1^{-1}$ if $y \ne z$.*

Proof. Assume that $y \neq z$; otherwise the result follows by Lemma 4.3(ii). Use the abbreviation $D_j^\ell = D_j^\ell(x, y)$ $(0 \leq \ell, j \leq D)$ and note that $z \in D_2^1$. It follows from Theorem 4.5 that the number of paths of length i between w and D_i^i is independent of x, y, z. Note that between any two vertices of Γ which are at distance $i-1$ (resp. i), there exist exactly $c_1 c_2 \cdots c_{i-1}(a_2 + a_3 + \cdots + a_{i-1})$ (resp. $c_1 c_2 \cdots c_i$) paths of length i. By Lemma 7.3 the number $|D_i^i \cap \Gamma_{i-1}(z)|$ is independent of x, y, z. Using these comments we find that $|D_i^i \cap \Gamma_i(z)|$ is independent of x, y, z. For $v \in D_i^i$ we have $|\Gamma_i(v) \cap \Gamma(x)| = a_i$ so $|\Gamma_i(v) \cap D_2^1| = a_i - 1$. Counting the number of pairs (w, v) such that $w \in D_2^1$, $v \in D_i^i$, $\partial(w, v) = i$ in two different ways and using the above comments we obtain $|D_i^i \cap \Gamma_i(z)||D_2^1| = |D_i^i|(a_i - 1)$. The result now follows by Lemmas 4.2(i) and 4.3. ■

Corollary 7.6. *With reference to Notation 3.1, for $1 \leq i \leq D$ we have*

$$E_1^* A_i E_i^* A_i E_1^* = a_i k_i k^{-1} E_1^* + a_i(a_i - 1) k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

Proof. For $z, w \in X$ we show that the (z, w)-entries of both sides are equal. If $z \notin \Gamma(x)$ or $w \notin \Gamma(x)$ then the (z, w)-entry of each side is 0. If $z, w \in \Gamma(x)$ then the (z, w)-entries of both sides are equal by Lemmas 5.1, 5.4 and 7.5. The result follows. ■

8. The irreducible T-modules with endpoint 1

With reference to Notation 3.1, in this section we describe the irreducible T-modules with endpoint 1.

Lemma 8.1. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1. Then $JW = 0$.*

Proof. Since W is not the primary module we have $E_0 W = 0$. Recall that $J = |X| E_0$ so $JW = 0$. ■

Lemma 8.2. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1. Then the following* (i)–(iii) *hold for $w \in E_1^* W$.*

(i) $E_1^* A w = 0$.
(ii) $E_i^* A_{i-1} w + E_i^* A_i w + E_i^* A_{i+1} w = 0$ *for* $1 \leq i \leq D - 1$.
(iii) $E_D^* A_{D-1} w + E_D^* A_D w = 0$.

Proof. For each equation in Corollary 5.3 apply both sides to w and simplify using $E_1^* w = w$ and Lemma 8.1. ■

Lemma 8.3. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1. Then the following* (i)–(iv) *hold for $w \in E_1^* W$ and $1 \leq i \leq D$.*

(i) $\|E_i^* A_{i-1} w\|^2 = c_i(k - c_i) k_i k^{-1} b_1^{-1} \|w\|^2$.
(ii) $\langle E_i^* A_{i-1} w, E_i^* A_i w \rangle = -a_i c_i k_i k^{-1} b_1^{-1} \|w\|^2$.
(iii) $\langle E_i^* A_i w, E_i^* A_{i-1} w \rangle = -a_i c_i k_i k^{-1} b_1^{-1} \|w\|^2$.
(iv) $\|E_i^* A_i w\|^2 = a_i(k - a_i) k_i k^{-1} b_1^{-1} \|w\|^2$.

Proof. (i) Evaluating $\|E_i^* A_{i-1} w\|^2$ using $E_1^* w = w$, line (1), and Corollary 7.2 we find

$$\|E_i^* A_{i-1} w\|^2 = \frac{c_i k_i}{k} \|w\|^2 + \frac{c_i (c_i - 1) k_i}{k b_1} \langle w, E_1^* A_2 w \rangle. \tag{6}$$

By Lemma 8.2(i),(ii) we have $E_1^* A_2 w = -w$. The result follows from this and (6).

(ii), (iii), (iv) Similar to the proof of (i) above. ■

Lemma 8.4. *With reference to Notation* 3.1 *let W denote an irreducible T-module with endpoint* 1. *Then for $w \in E_1^* W$ and $2 \le i \le D - 1$ the determinant of*

$$\begin{pmatrix} \|E_i^* A_{i-1} w\|^2 & \langle E_i^* A_{i-1} w, E_i^* A_i w \rangle \\ \langle E_i^* A_i w, E_i^* A_{i-1} w \rangle & \|E_i^* A_i w\|^2 \end{pmatrix}$$

is equal to

$$a_i b_i c_i k_i^2 k^{-1} b_1^{-2} \|w\|^4.$$

Proof. Evaluate the matrix entries using Lemma 8.3 and take the determinant. ■

Theorem 8.5. *With reference to Notation* 3.1 *let W denote an irreducible T-module with endpoint* 1 *and fix a nonzero $w \in E_1^* W$. Then the following is a basis for W:*

$$E_i^* A_{i-1} w, \qquad E_{i+1}^* A_{i+1} w \quad (1 \le i \le D - 1). \tag{7}$$

Proof. We first show that W is spanned by the vectors (7). Let W' denote the subspace of V spanned by the vectors (7) and note that $W' \subseteq W$. We claim that W' is a T-module. By construction W' is M^*-invariant. It follows from Lemmas 6.2–6.4 and 8.2 and $E_1^* w = w$ that W' is invariant under each of L, F, R. Recall that $L + F + R = A$ and A generates M so W' is M-invariant. The claim follows. Note that $W' \neq 0$ since $w \in W'$ so $W' = W$ by the irreducibility of W.

We now show that the vectors (7) are linearly independent. Because of (5) it suffices to show: $w \neq 0$, the vectors $E_i^* A_{i-1} w$, $E_i^* A_i w$ are linearly independent for $2 \le i \le D - 1$, and $E_D^* A_D w \neq 0$. First note that $w \neq 0$ by assumption and $E_D^* A_D w \neq 0$ by Lemma 8.3(iv). For $2 \le i \le D - 1$ the vectors $E_i^* A_{i-1} w$, $E_i^* A_i w$ are linearly independent since their matrix of inner products has nonzero determinant by Lemma 8.4. ■

We emphasize an idea from the proof of Theorem 8.5.

Corollary 8.6. *With reference to Notation* 3.1 *let W denote an irreducible T-module with endpoint* 1. *Then the following* (i)–(iii) *hold for a nonzero $w \in E_1^* W$.*

(i) *$E_1^* W$ has a basis*

$$w.$$

(ii) *For $2 \le i \le D - 1$ the subspace $E_i^* W$ has a basis*

$$E_i^* A_{i-1} w, \qquad E_i^* A_i w.$$

(iii) *$E_D^* W$ has a basis*

$$E_D^* A_D w.$$

Remark 8.7. With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and fix a nonzero $w \in E_1^*W$. It follows from Lemma 8.3(ii) that for $2 \le i \le D-1$ the basis for E_i^*W given in Corollary 8.6(ii) is not orthogonal. However, using the Gram–Schmidt orthogonalization one finds that $E_i^*A_{i-1}w, E_i^*A_{i-1}w + (k - c_i)a_i^{-1}E_i^*A_iw$ is an orthogonal basis for E_i^*W.

Corollary 8.8. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1. Then the following (i)–(iii) hold.*

(i) *E_1^*W has dimension 1.*
(ii) *E_i^*W has dimension 2 for $2 \le i \le D-1$.*
(iii) *E_D^*W has dimension 1.*

Proof. Immediate from Corollary 8.6. ■

Corollary 8.9. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1. Then the following (i), (ii) hold.*

(i) *The dimension of W is $2D-2$.*
(ii) *The diameter of W is $D-1$.*

Proof. Immediate from Corollary 8.8. ■

Corollary 8.10. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and fix a nonzero $w \in E_1^*W$. Then $W = M^*Mw$.*

Proof. By construction $M^*Mw \subseteq W$ and equality holds in view of Theorem 8.5. ■

9. Irreducible T-modules with endpoint 1: the A-action

With reference to Notation 3.1 let W denote irreducible T-module with endpoint 1. In this section we display the action of A on the basis for W given in Theorem 8.5. Since $A = L+F+R$ it suffices to give the actions of L, F, R on this basis.

Lemma 9.1. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and fix a nonzero $w \in E_1^*W$. Then the matrix L from Definition 6.1 satisfies the following (i)–(v).*

(i) $Lw = 0$.
(ii) $LE_2^*Aw = (k - c_2)w$.
(iii) $LE_i^*A_{i-1}w = (b_{i-1}+c_{i-1}+\rho_i-c_i)E_{i-1}^*A_{i-2}w+(b_{i-1}+c_{i-1}+\rho_i-c_i-\tau_{i-1})E_{i-1}^*A_{i-1}w$ *for* $3 \le i \le D-1$.
(iv) $LE_2^*A_2w = -a_2w$.
(v) $LE_i^*A_iw = (a_{i-1}-a_i-\rho_i)E_{i-1}^*A_{i-2}w+(\tau_{i-1}+a_{i-1}-a_i-\rho_i)E_{i-1}^*A_{i-1}w$ *for* $3 \le i \le D$.

Proof. For each equation of Lemma 6.2, apply each side to w and simplify using $E_1^*w = w$ and Lemma 8.2. ■

Lemma 9.2. *With reference to Notation 3.1 let W denote an irreducible T-module with endpoint 1 and fix a nonzero $w \in E_1^*W$. Then the matrix F from Definition 6.1 satisfies the following (i)–(iv).*

(i) $Fw = 0$.
(ii) $FE_i^*A_{i-1}w = (a_{i-1} - \rho_i)E_i^*A_{i-1}w + (c_i - \sigma_i)E_i^*A_iw$ *for* $2 \le i \le D-1$.
(iii) $FE_i^*A_iw = (a_i - a_{i-1} + \rho_i - \rho_{i+1})E_i^*A_{i-1}w + (a_i - b_i - c_i + \sigma_i + \tau_i - \rho_{i+1})E_i^*A_iw$ *for* $2 \le i \le D-1$.
(iv) $FE_D^*A_Dw = (a_{D-1} - c_D + \sigma_D - \rho_D)E_D^*A_Dw$.

Proof. For each equation of Lemma 6.3, apply each side to w and simplify using $E_1^*w = w$ and Lemma 8.2. ■

Lemma 9.3. *With reference to Notation* 3.1 *let W denote an irreducible T-module with endpoint* 1 *and fix a nonzero $w \in E_1^*W$. Then the matrix R from Definition* 6.1 *satisfies the following* (i)–(v).

(i) $RE_i^*A_{i-1}w = c_iE_{i+1}^*A_iw$ *for* $1 \le i \le D-2$.
(ii) $RE_{D-1}^*A_{D-2}w = -c_{D-1}E_D^*A_Dw$.
(iii) $RE_i^*A_iw = \rho_{i+1}E_{i+1}^*A_iw + \sigma_{i+1}E_{i+1}^*A_{i+1}w$ *for* $2 \le i \le D-2$.
(iv) $RE_{D-1}^*A_{D-1}w = (\sigma_D - \rho_D)E_D^*A_Dw$.
(v) $RE_D^*A_Dw = 0$.

Proof. For each equation of Lemma 6.4, apply each side to w and simplify using $E_1^*w = w$ and Lemma 8.2. ■

10. The isomorphism class of irreducible T-modules with endpoint 1

With reference to Notation 3.1, in this section we prove that up to an isomorphism there exists a unique irreducible T-module with endpoint 1.

Theorem 10.1. *With reference to Notation* 3.1*, any two irreducible T-modules with endpoint* 1 *are isomorphic.*

Proof. Let W and W' denote irreducible T-modules with endpoint 1 and fix nonzero $w \in E_1^*W$, $w' \in E_1^*W'$. By Theorem 8.5, W and W' have bases $\{E_i^*A_{i-1}w, E_{i+1}^*A_{i+1}w \mid 1 \le i \le D-1\}$ and $\{E_i^*A_{i-1}w', E_{i+1}^*A_{i+1}w' \mid 1 \le i \le D-1\}$, respectively. Let $\sigma : W \to W'$ denote the vector space isomorphism defined by $\sigma(E_i^*A_{i-1}w) = E_i^*A_{i-1}w'$ and $\sigma(E_{i+1}^*A_{i+1}w) = E_{i+1}^*A_{i+1}w'$ for $1 \le i \le D-1$. We show that σ is a T-module isomorphism. Since A generates M and $E_0^*, E_1^*, \ldots, E_D^*$ is a basis for M^*, it suffices to show that σ commutes with each of $A, E_0^*, E_1^*, \ldots, E_D^*$.

Using the assertion (iv) below the line (4) and the definition of σ we immediately find that σ commutes with each of $E_0^*, E_1^*, \ldots, E_D^*$. It follows from Lemmas 9.1–9.3 that σ commutes with each of L, F, R. Recall that $A = L + F + R$ so σ commutes with A. The result follows. ■

With reference to Notation 3.1 recall that V is an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity with which W appears in V* we mean the number of irreducible T-modules in this sum which are isomorphic to W. For example the primary module $M\hat{x}$ appears in V with multiplicity 1.

Corollary 10.2. *With reference to Notation* 3.1*, let W denote an irreducible T-module with endpoint* 1*. Then W appears in V with multiplicity $k-1$.*

Proof. Write V as an orthogonal direct sum of irreducible T-modules $W_0, W_1, \ldots, W_\ell$. Without loss of generality we assume that $W_0 = M\hat{x}$. Note that $E_1^*V = \sum_{i=0}^{\ell} E_1^*W_i$ (direct sum) so

$$\dim E_1^*V = \sum_{i=0}^{\ell} \dim E_1^*W_i. \tag{8}$$

We examine the terms in (8). We mentioned earlier that $\dim E_1^*V = k$. By [7, Theorem 5.4] $E_1^*W_0$ is spanned by $\sum_{y\in\Gamma(x)} \hat{y}$ so $\dim E_1^*W_0 = 1$. By Corollary 8.8 and construction, for $1 \le i \le \ell$ the dimension of $E_1^*W_i$ is 1 (resp. 0) if the endpoint of W_i is 1 (resp. at least 2). Evaluating (8) using these comments we find that there exist exactly $k-1$ T-modules among $W_0, W_1, \ldots, W_\ell$ that have endpoint 1. The result follows in view of Theorem 10.1. ■

Corollary 10.3. *With reference to Notation 3.1 fix a nonzero $w \in E_1^*V$ which is orthogonal to $\sum_{y\in\Gamma(x)} \hat{y}$. Then M^*Mw is an irreducible T-module with endpoint* 1.

Proof. Let H denote the subspace of V spanned by the irreducible T-modules with endpoint 1. By construction and Lemma 8.1 E_1^*H is the orthogonal complement of $\sum_{y\in\Gamma(x)} \hat{y}$ in E_1^*V. Hence $w \in E_1^*H$. Note that $Tw \subseteq H$ so Tw is the orthogonal direct sum of some irreducible T-modules of endpoint 1. Call these T-modules $W_1, W_2, \ldots, W_s$. Since $w \in Tw$ we have $w = w_1 + w_2 + \cdots + w_s$ for some $w_i \in E_1^*W_i$ $(1 \le i \le s)$. Without loss of generality we may assume that $w_1 \ne 0$. Since $w_1 \in Tw$ there exists $t \in T$ such that $w_1 = tw$. But now $w_1 = tw = tw_1 + tw_2 + \cdots + tw_s$, forcing $tw_i = 0$ for $2 \le i \le s$ and $tw_1 = w_1 \ne 0$. But since $W_1, W_2, \ldots, W_s$ are mutually isomorphic by Theorem 10.1 this is a contradiction unless $w_2 = \cdots = w_s = 0$. Therefore $w = w_1$ and $Tw = W_1$. It follows that Tw is an irreducible T-module with endpoint 1. The result follows since $Tw = M^*Mw$ by Corollary 8.10. ■

11. The case of classical parameters

The distance-regular graph Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (1 \le i \le D), \tag{9}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \le i \le D-1), \tag{10}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}.$$

In this case b is an integer and $b \notin \{0, -1\}$ [2, Proposition 6.2.1]. Assume that Γ has classical parameters (D, b, α, β). By (9), (10) and $a_i = k - b_i - c_i$ we find

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(\beta - 1 + \alpha\left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \qquad (1 \le i \le D). \tag{11}$$

In particular

$$a_1 = \beta - 1 + \alpha b\begin{bmatrix} D-1 \\ 1 \end{bmatrix}. \tag{12}$$

By [2, Corollary 8.4.2] Γ has a Q-polynomial structure such that

$$b(\theta_1^* - \theta_i^*) = \theta_0^* - \theta_{i-1}^* \quad (1 \le i \le D). \tag{13}$$

Pick an integer i $(2 \le i \le D)$. By a *parallelogram of length* i in Γ we mean a 4-tuple $uvzw$ of vertices of Γ such that $\partial(u, v) = 1$, $\partial(z, w) = 1$, $\partial(u, z) = i$, $\partial(u, w) = i - 1$, $\partial(v, w) = i - 1$, $\partial(v, z) = i - 1$.

Theorem 11.1. *With reference to Notation 3.1 the following* (i)–(iii) *are equivalent.*

(i) *Γ has classical parameters (D, b, α, β).*
(ii) *$\rho_i = 0$ for $3 \le i \le D$, where ρ_i is from Theorem 4.5.*
(iii) *Γ contains no parallelograms of length i for $3 \le i \le D$.*

Suppose (i)–(iii) *hold. Then $\beta = 1 - \alpha b \left[{D-1 \atop 1} \right]$. Moreover $b < -1$.*

Proof. (i) $\Rightarrow$ (ii) For $3 \le i \le D$ we evaluate the expression for ρ_i in Theorem 4.5 using (13) and routinely find $\rho_i = 0$.

(ii) $\Rightarrow$ (iii) Assume that there exists a parallelogram of some length i $(3 \le i \le D)$. Since x was chosen arbitrarily we may assume that this parallelogram is of the form $xyzw$ for some $y, z, w \in X$. Use the abbreviation $D_j^\ell = D_j^\ell(x, y)$ for $0 \le j, \ell \le D$ and note that $w \in D_{i-1}^{i-1}$, $z \in D_{i-1}^i$. But then $\rho_i \neq 0$ in view of Theorem 4.5, a contradiction.

(iii) $\Rightarrow$ (i) This follows from [24, Theorem 1.1]. ■

Suppose (i)–(iii) hold. Then $\beta = 1 - \alpha b \left[{D-1 \atop 1} \right]$ by (12) and since $a_1 = 0$. Now $b < -1$ by [24, Lemma 3.3]. ■

Theorem 11.2. *With reference to Notation 3.1 assume that Γ has classical parameters (D, b, α, β). Then the following* (i), (ii) *hold.*

(i) *The intersection numbers a_i, b_i, c_i of Γ satisfy*

$$a_i = -\frac{\alpha(b+1)(b^i - 1)(b^{i-1} - 1)}{(b-1)^2},$$
$$b_i = \frac{(b^D - b^i)(b - 1 - \alpha(b^D + b^i - b - 1))}{(b-1)^2},$$
$$c_i = \frac{(b^i - 1)(\alpha b^{i-1} + b - \alpha - 1)}{(b-1)^2} \quad (0 \le i \le D).$$

(ii) *The integers ρ_i, σ_i, τ_i from Theorem 4.5 satisfy*

$$\rho_i = 0, \qquad \sigma_i = c_{i-1} - b^{i-2}, \qquad \tau_i = b_i, \quad (2 \le i \le D).$$

Proof. (i) Evaluate a_i, b_i, c_i using (11), (10), (9) and $\beta = 1 - \alpha b \left[{D-1 \atop 1} \right]$.

(ii) Note that $a_1 = 0$ implies $\rho_2 = 0$ and that for $3 \le i \le D$ we have $\rho_i = 0$ by Theorem 11.1. Evaluating the expressions for σ_i, τ_i in Theorem 4.5 using (13) we routinely find $\tau_i = b_i$ and $\sigma_i / c_i = (b^{i-2} - 1)/(b^i - 1)$. The result now follows using (i) above. ■

We finish the paper with two examples.

Example 11.3. The *Hermitian forms graph* has classical parameters (D, b, α, β), where $b = -r$, r a prime power, $\alpha = b - 1$, $\beta = -b^D - 1$ [2, p. 194, Table 6.1]. Using (12) we find $a_1 = 0$ if and only if $r = 2$. Assume that $r = 2$. Then using Theorem 11.2 we find

$$a_i = \frac{2^{2i-1} - (-2)^{i-1} - 1}{3}, \qquad b_i = \frac{4^D - 4^i}{3}, \qquad c_i = \frac{(-2)^{i-1}(1 - (-2)^i)}{3}$$

for $0 \le i \le D$ and

$$\rho_i = 0, \qquad \sigma_i = \frac{(-2)^{i-1}(1 - (-2)^{i-2})}{3}, \qquad \tau_i = \frac{4^D - 4^i}{3} \quad (2 \le i \le D).$$

Example 11.4. The *Witt graph* M_{23} has classical parameters $(3, -2, -2, 5)$ [2, p. 194, Table 6.1]. Using Theorem 11.2 we find

$$a_1 = 0,\ a_2 = 2,\ a_3 = 6, \qquad b_0 = 15,\ b_1 = 14,\ b_2 = 12, \qquad c_1 = 1,\ c_2 = 1,\ c_3 = 9$$

and

$$\rho_2 = 0,\ \rho_3 = 0, \qquad \sigma_2 = 0,\ \sigma_3 = 3, \qquad \tau_2 = 12,\ \tau_3 = 0.$$

Remark 11.5. Beside Examples 11.3 and 11.4 we are not aware of any Q-polynomial distance-regular graph with diameter $D \ge 3$ such that $a_1 = 0$ and $a_i \ne 0$ for $2 \le i \le D$.

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References

[1] E. Bannai, T. Ito, Algebraic Combinatorics I: Association schemes, in: Benjamin-Cummings Lecture Notes, vol. 58, Menlo Park, 1984.
[2] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, Heidelberg, 1989.
[3] J. Caughman, The Terwilliger algebras of bipartite P- and Q-polynomial association schemes, Discrete Math. 196 (1999) 65–95.
[4] J. Caughman, M. MacLean, P. Terwilliger, The Terwilliger algebra of an almost-bipartite P- and Q-polynomial scheme, Discrete Math. 292 (2005) 17–44.
[5] B. Collins, The girth of a thin distance-regular graph, Graphs Combin. 13 (1997) 21–30.
[6] B. Collins, The Terwilliger algebra of an almost-bipartite distance-regular graph and its antipodal 2-cover, Discrete Math. 216 (2000) 35–69.
[7] B. Curtin, Bipartite distance-regular graphs, Part I, Graphs Combin. 15 (1999) 143–158.
[8] B. Curtin, Bipartite distance-regular graphs, Part II, Graphs Combin. 15 (1999) 377–391.
[9] B. Curtin, The local structure of a bipartite distance-regular graph, European J. Combin. 20 (1999) 739–758.
[10] G. Dickie, Twice Q-polynomial distance-regular graphs are thin, European J. Combin. 16 (1995) 555–560.
[11] G. Dickie, P. Terwilliger, A note on thin P-polynomial and dual-thin Q-polynomial symmetric association schemes, J. Algebraic Combin. 7 (1998) 5–15.
[12] E. Egge, A generalization of the Terwilliger algebra, J. Algebra 233 (2000) 213–252.

[13] E. Egge, The generalized Terwilliger algebra and its finite-dimensional modules when $d = 2$, J. Algebra 250 (2002) 178–216.
[14] J. Go, The Terwilliger algebra of the hypercube, European J. Combin. 23 (2002) 399–429.
[15] J. Go, Tight distance-regular graphs and the subconstituent algebra, European J. Combin. 23 (2002) 793–816.
[16] S.A. Hobart, T. Ito, The structure of nonthin irreducible T-modules of endpoint 1: Ladder bases and classical parameters, J. Algebraic Combin. 7 (1998) 53–75.
[17] T. Ito, K. Tanabe, P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp. 167–192.
[18] T. Ito, P. Terwilliger, The shape of a tridiagonal pair, J. Pure Appl. Algebra 188 (2004) 145–160.
[19] T. Ito, P. Terwilliger, Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$, Ramanujan J. 13 (2007) 41–64.
[20] T. Ito, P. Terwilliger, Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations, J. Algebra Appl. 6 (2007) 477–503.
[21] Š Miklavič, Q-polynomial distance-regular graphs with $a_1 = 0$, European J. Combin. 25 (2004) 911–920.
[22] K. Nomura, A refinement of the split decomposition of a tridiagonal pair, Linear Algebra Appl. 403 (2005) 1–23.
[23] K. Nomura, P. Terwilliger, The split decomposition of a tridiagonal pair, Linear Algebra Appl. 424 (2007) 339–345.
[24] Y. Pan, M. Lu, C. Weng, Triangle-free Distance-regular Graphs, J. Algebraic Combin. 27 (2008) 23–34.
[25] K. Tanabe, The irreducible modules of the Terwilliger algebras of Doob schemes, J. Algebraic Combin. 6 (1997) 173–195.
[26] P. Terwilliger, The subconstituent algebra of an association scheme, J. Algebraic Combin. Part I 1 (1992) 363–388. Part II: 2 (1993), 73-103; Part III: 2 (1993), 177-210.
[27] M. Tomiyama, N. Yamazaki, The subconstituent algebra of a strongly regular graph, Kyushu J. Mat. 48 (1998) 323–334.